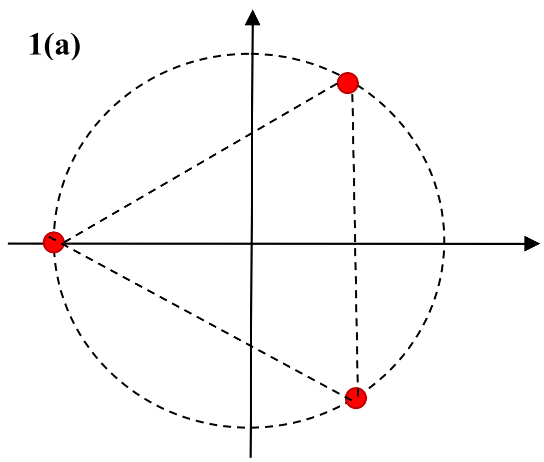
Math 332 • Midterm Exam • March 9, 2016 • Victor Matveev

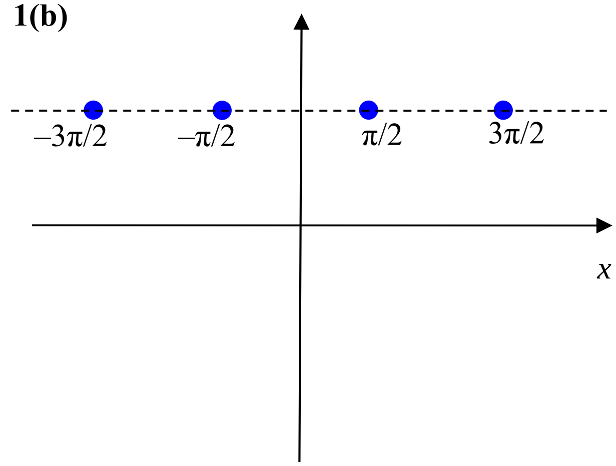
1) **(24pts)** Find **all distinct** values of *z*, in Cartesian or polar form. For parts (a) and (b), show the locations of these points in the complex plane

$$(a)\quad z=(1-i)^{4/3}=\left(\sqrt{2}\,e^{-i\frac{\pi}{3}+i2\pi n}\right)^{4/3}=\sqrt[3]{4}\,e^{-i\frac{\pi}{3}+i\frac{8\pi n}{3}}=\begin{cases} n=0: & \boxed{\sqrt[3]{4}\,e^{-i\frac{\pi}{3}}} \\ n=1: & \sqrt[3]{4}\,e^{-i\frac{\pi}{3}+i\frac{8\pi}{3}}=\sqrt[3]{4}\,e^{i\frac{7\pi}{3}}=\boxed{\sqrt[3]{4}\,e^{i\frac{\pi}{3}}} \\ n=-1: & \sqrt[3]{4}\,e^{-i\frac{\pi}{3}-i\frac{8\pi}{3}}=\sqrt[3]{4}\,e^{-3i\pi}=\boxed{-\sqrt[3]{4}} \end{cases}$$

1(a)



1(b)



$$(b)\quad \tanh z=2i \;\Rightarrow\; \frac{\sin z}{\cos z}=\frac{1}{i}\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}=2i \;\Rightarrow\; e^{iz}-e^{-iz}=-2\left(e^{iz}+e^{-iz}\right)\Rightarrow\; e^{2iz}-1=-2\left(e^{2iz}+1\right)$$

$$\Rightarrow\; 3e^{2iz}=-1 \;\Rightarrow\; e^{2iz}=-\frac{1}{3} \;\Rightarrow\; 2iz=\log\left(-\frac{1}{3}\right)=\ln\left(\frac{1}{3}\right)+i\pi+i2\pi n=-\ln 3+i\pi(1+2n)$$

$$\Rightarrow\; z=\frac{1}{2i}\left(-\ln 3+i\pi(1+2n)\right) \;\Rightarrow\; \boxed{z=\pi\left(\frac{1}{2}+n\right)+i\frac{\ln 3}{2}} \quad n\in\mathbb{Z}$$

$$(c)\quad z=(-i)^{1-i}=\left(e^{-i\frac{\pi}{2}+i2\pi n}\right)^{1-i}=e^{\left(-i\frac{\pi}{2}+i2\pi n\right)(1-i)}=\underbrace{e^{-\frac{\pi}{2}+2\pi n}}_{|z|}\;\underbrace{e^{i\pi\left(2n-\frac{1}{2}\right)}}_{=e^{i\pi/2}=i}=\boxed{i\,e^{-\frac{\pi}{2}+2\pi n}}$$

2) (32pts) Calculate each integral over the given circle, or explain *clearly* why the integral equals zero; make sure to indicate the locations of singularities of each integrand:

$$(a)\quad \oint_{|z|=5} \frac{e^z\,dz}{\left(e^z-1\right)^9}=0 \quad \text{by F.C.T.: antiderivative is continuous along the entire contour } F(z)=\frac{1}{\left(e^z-1\right)^8}$$

(Singularitis are at $\log(1)=i2\pi n$, none of which are on the contour)

$$(b)\quad \oint_{|z|=1} \frac{dz}{\cos z+1}=0 \quad \text{by C.G.T. since the nearest singularity } (\cos z=-1) \text{ is at } z=\pm\pi, \text{ outside the circle } |z|=1$$

$$(c)\quad \oint_{|z|=2} \frac{\sin\left(z^3\right)dz}{z^2+1} = \oint_{|z|=2} \frac{\sin\left(z^3\right)dz}{(z+i)(z-i)} = \oint_{|z+i|=\varepsilon} \frac{\sin\left(z^3\right)/(z-i)\,dz}{z+i} + \oint_{|z+i|=\varepsilon} \frac{\sin\left(z^3\right)/(z+i)\,dz}{z-i}$$

$$=2\pi i\left[\frac{\sin(-i)^3}{-i-i}+\frac{\sin(i)^3}{i+i}\right]=2\pi i\left[\frac{\sin(+i)}{-2i}+\frac{\sin(-i)}{2i}\right]=-2\pi\sin i=\boxed{-2\pi i\sinh 1}$$

$$d)\quad \oint_{|z|=4} \frac{dz}{\sqrt{z}}\neq 0 \Leftarrow \text{ Antiderivative } F(z)=2\sqrt{z} \text{ has a discontinuity (sign change) on the contour, at } z=-4\text{:}$$

thus, the integral equals the jump: $\displaystyle \oint_{|z|=4} \frac{dz}{\sqrt{z}}=\left[2\sqrt{z}\right]_{4e^{-i\pi}}^{4e^{+i\pi}}=2\left(\sqrt{4e^{i\pi}}-\sqrt{4e^{-i\pi}}\right)=4\left(i-(-i)\right)=\boxed{8i}$

Alternatively, you can compute this directly by parametrizing the circle: $z=4e^{i\theta},\ \theta\in[-\pi,\pi]$

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3) (14pts) Differentiate this function: $f(z)=(\cos z)^{\operatorname{Log} z}$

$$(\cos z)^{\operatorname{Log} z}=e^{\operatorname{Log}(\cos z)\operatorname{Log} z}$$

$$\Rightarrow \frac{df}{dz}=e^{\operatorname{Log}(\cos z)\operatorname{Log} z}\frac{d}{dz}\left[\operatorname{Log}(\cos z)\operatorname{Log} z\right]$$

$$=(\cos z)^{\operatorname{Log} z}\left[\frac{-\sin z}{\cos z}\operatorname{Log} z+\frac{\operatorname{Log}(\cos z)}{z}\right]=\boxed{(\cos z)^{\operatorname{Log} z}\left[\frac{\operatorname{Log}(\cos z)}{z}-\tan z\operatorname{Log} z\right]}$$

4) (14pts) Is the function $\overline{z}$ $f(z)=\dfrac{(\overline{z})^2}{z}$ differentiable anywhere? Is it analytic anywhere? Is this function continuous in the entire plane? Use one of the following forms of Cauchy-Riemann equations in polar coordinates to analyze analyticity / differentiability:

$$\frac{df}{dz}=e^{-i\theta}\frac{\partial f}{\partial r}=-i\frac{e^{-i\theta}}{r}\frac{\partial f}{\partial\theta} \Rightarrow \text{ or, written in component form} \Rightarrow \begin{cases} u_r=\dfrac{v_\theta}{r} \\ v_r=-\dfrac{u_\theta}{r} \end{cases}$$

A function explicitly dependent on $\overline{z}$ can't be analytic, which is easy to see in polar coordinates:

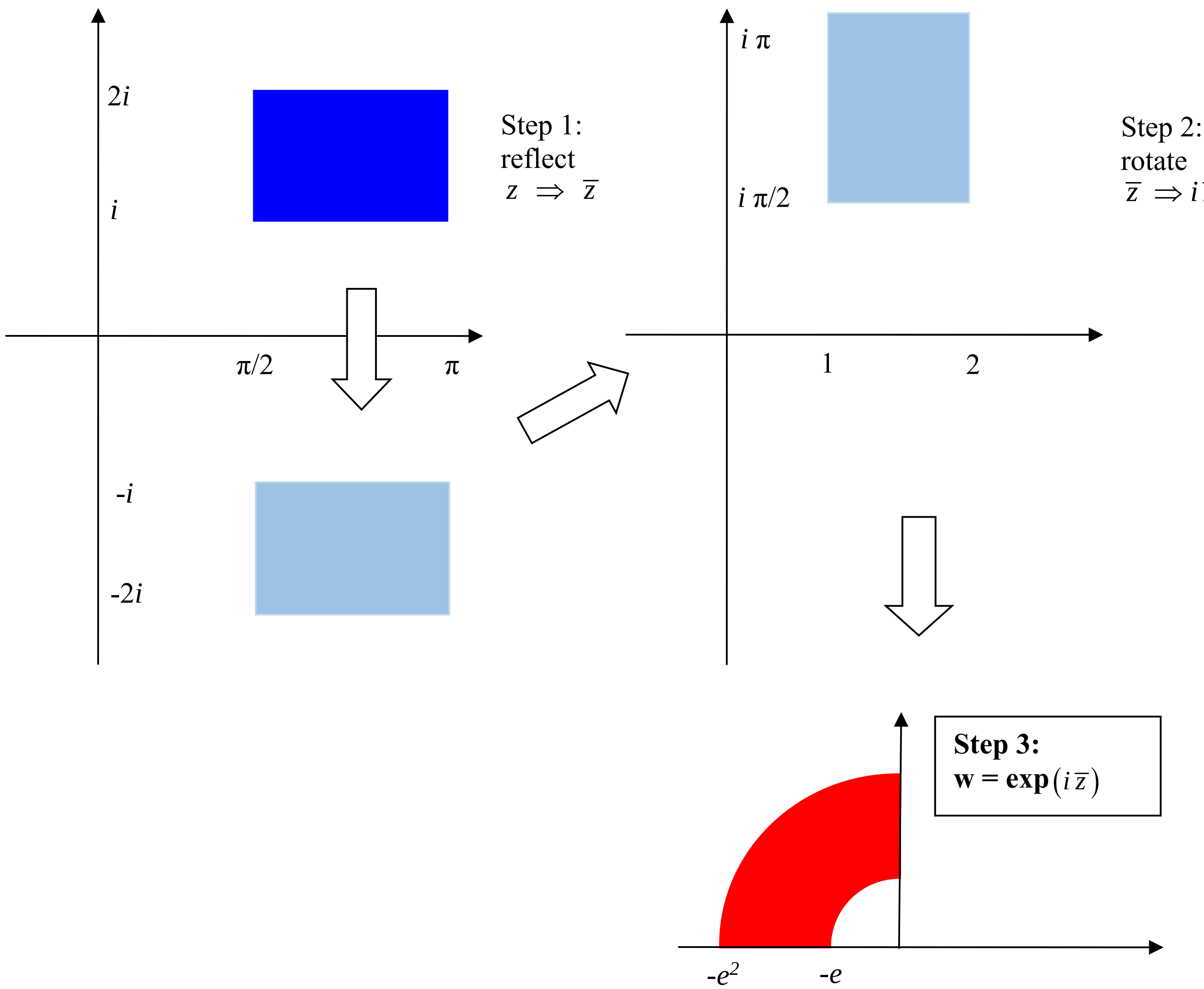
$$f(z)=\frac{(\overline{z})^2}{z}=\frac{\left(re^{-i\theta}\right)^2}{re^{i\theta}}=re^{-3i\theta} \Rightarrow \begin{cases} \dfrac{\partial f}{\partial r}=e^{-3i\theta} \\ \dfrac{-i}{r}\dfrac{\partial f}{\partial\theta}=-3e^{-3i\theta} \end{cases} \quad \text{Not equal anywhere}$$

Thus, the function is neither differentiable nor analytic anywhere

However, it can be made continuous in the entire plane by defining $f(0)=0$, which removes the removable discontinuity at z=0 since it has a limit at z=0: $\lim\limits_{z\to 0}\left[f(z)\right]=\lim\limits_{r\to 0}\left[re^{-3i\theta}\right]=0$

============= Pick 1 problem out of the last 2 (i.e. drop one problem) =========

5) **(16pts)** Sketch the region $\pi/2 \le \text{Re } z \le \pi$, $1 \le \text{Im } z \le 2$, and sketch its image under the transformation $w = \exp(i\bar{z})$. It may help to decompose this map into three elementary steps.



Step 1: reflect around the real axis:

Step 2: rotate counterclockwise by $\pi/2$: $\text{Re}(w = i\bar{z}) \in [1, 2]$; $\text{Im}(w = i\bar{z}) \in \left[\frac{\pi}{2}, \pi\right]$

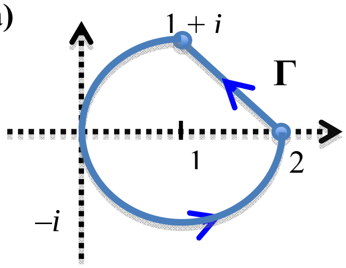
Step 3: exponentiate: $\exp w = \exp(u + iv) = \exp(u)\exp(iv)$

6) (16pts) Calculate the following integrals, using an appropriate method in each case, or explain why the integral is zero:

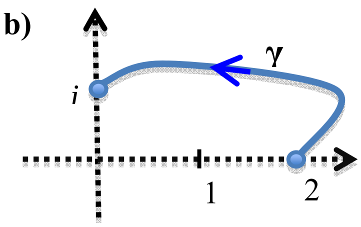
a) $\oint_\Gamma \mathrm{Im}(z)\,dz$, where Γ is shown in the top figure

b) $\int_\gamma \frac{z\,dz}{\left(z^2-1\right)^2}$, where γ is shown in the bottom figure

a)



b)



$$(a)\quad \oint_\Gamma \mathrm{Im}(z)\,dz = \int_{\substack{\text{Line:}\\ z=2+(i-1)t\\ t\in[0,1]}} \underbrace{\mathrm{Im}(z)}_{t}\,\underbrace{dz}_{(i-1)dt} + \int_{\substack{\text{Circle:}\\ z=1+\exp(it)\\ t\in\left[\frac{\pi}{2},2\pi\right]}} \underbrace{\mathrm{Im}(z)}_{\sin t}\,\underbrace{dz}_{ie^{it}dt}$$

$$= (i-1)\underbrace{\int_0^1 t\,dt}_{1/2} + \int_{\pi/2}^{2\pi} \underbrace{\frac{e^{it}-e^{-it}}{2i}ie^{it}}_{\left(e^{2it}-1\right)/2}\,dt = \frac{i-1}{2}+\frac{1}{2}\left[\frac{e^{2it}}{2i}-t\right]_{\pi/2}^{2\pi} = \frac{i-1}{2}+\frac{1}{2}\left[-i-\frac{3\pi}{2}\right] = \boxed{-\frac{1}{2}-\frac{3\pi}{4}}$$

Note that the absolute value equals the area enclosed by the contour, in agreement with Green's Theorem

$$\int_\gamma \frac{z\,dz}{\left(z^2-1\right)^2} = \left[-\frac{1}{2\left(z^2-1\right)}\right]_2^i = -\frac{1}{2(-1-1)}+\frac{1}{2(4-1)} = \frac{1}{4}+\frac{1}{6} = \boxed{\frac{5}{12}}$$